Geometry of Two-Qubit State and Intertwining Quaternionic Conformal Mapping Under Local Unitary Transformations

G. Najarbashi [a,b] *, S. Ahadpour [a,b] †, M. A. Fasihi [c] ‡, Y. Tavakoli [a]§

[a]Department of Theoretical Physics and Astrophysics, Tabriz University, Tabriz 51664, Iran.

[b]Department of Physics, Mohaghegh Ardabili University, Ardabil 56199-11367, Iran.

[c]Department of Physics, Azarbaijan University of Tarbiat Moallem, Tabriz 53714-161, Iran.

February 1, 2008

*E-mail:najarbashi@tabrizu.ac.ir

†E-mail:ahadpour@uma.ac.ir

‡E-mail:ma-fasihi@azaruniv.edu

§E-mail:tavakoli_ phys@yahoo.com

Abstract

In this paper the geometry of two-qubit systems under local unitary group $SO(2) \otimes SU(2)$ is discussed. It is shown that the quaternionic conformal map intertwines between this local unitary subgroup of $Sp(2)$ and the quaternionic Möbius transformation which is rather a generalization of the results of Lee et al (2002 Quantum Inf. Process. 1 129).

Keywords: Conformal map, Quaternion, Entanglement, Möbius Transformation.

PACs Index: 03.67.a 03.65.Ud

1 Introduction

There has been considerable recent interest in understanding the structure of one,two, three and multi-qubit systems, from the geometrical point of view [1, 2, 3, 4, 5, 6, 7, 8]. The relation between *conformal map* (or Hopf fibration in [2]) and single qubit and two-qubit states have first been studied by Mosseri and Dandoloff [2] in quaternionic skew-field and subsequently have been generalized to three-qubit state based on octonions by Bernevig and Chen [6]. Also some attempts have been made to figure out the notion of entanglement and basic geometry of the space of states [5, 7, 8]. From an information-theoretic standpoint, the construction of well-defined entanglement measure typically relies on the concept of entanglement monotone which is non-increasing under local operations and classical communication. Such transformations are called LOCC [4, 9, 10]. For instance, the most widely utilized measure for two-qubit, is concurrence introduced by Wootters [11].

However it seems that there is also another geometrical approach to describe pure two-qubit states called conformal groups [12]. As is typical in physics, the local properties are more immediately useful than the global properties, and the local unitary transformation is of great importance. Therefore in this paper we pursue a different approach to study the geometrical structure of two-qubit states under local unitary subgroup of $Sp(2)$ [13]. We show that the quaternionic conformal map (QCM) of a pure two-qubit system intertwines between the local subgroup $Sp(2)$ and corresponding *quaternionic Möbius transformations* (QMT) [14, 15, 16] which can be useful in theoretical physics such as quaternionic quantum mechanics [17], quantum conformal field theory [12, 18] and quaternionic computations [19]. However the action of transformations that involve with non-commutative quaternionic skew-field on a spinor (living in quaternionic Hilbert spaces) is more complicated than the complex one. Roughly speaking one must distinguish between the left and right actions of a quaternionic transformations on a given state (e.g see [20]). This anomalous property of quaternionic

transformation lead us to define the special QMT.

The paper is organized as follows: In Section 2. we briefly summarize one-qubit geometry and conformal map in a commutative diagram. In section 3, we introduce the basic geometric structure together with basic background material, incorporating all the information we need for characterization of two-qubit geometry. Section 4 devoted to study the commutativity of QCM in details. The paper is ended with a brief conclusion and one appendix.

2 One-qubit geometry

We will denote by $\mathcal{H}_d^{\mathbb{F}}$ the Hilbert space of dimension d in $\mathbb{F} = \mathbb{C}$ or $\mathbb{Q}$. Let us consider an arbitrary one-qubit pure state in complex two dimensional Hilbert space $\mathcal{H}_2^{\mathbb{C}}$

$$|\psi\rangle = \alpha_1|0\rangle + \alpha_2|1\rangle \qquad , \quad |\alpha_1|^2 + |\alpha_2|^2 = 1 \qquad , \quad \alpha_1, \alpha_2 \in \mathbb{C}. \tag{1}$$

We summarize the results of Ref.[1] in a commutative diagram fashion convenient for our purposes as:

$$\begin{array}{ccc} \mathcal{H}_2^{\mathbb{C}} & \xrightarrow{\mathcal{P}} & \widetilde{\mathbb{C}} \\ {\scriptstyle A}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{F}_A} \\ \mathcal{H}_2^{\mathbb{C}} & \xrightarrow[\mathcal{P}]{} & \widetilde{\mathbb{C}} \end{array}$$

where $\mathcal{P}$ is conformal mapping for one-qubit system, i.e.,

$$\mathcal{P}(|\psi\rangle) := \alpha_1\alpha_2^{-1} \in \widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}, \tag{2}$$

and $\mathcal{F}_A \in PSU(2) = SU(2)/\{\pm I\}$ is Möbius transformation corresponding to 2×2 matrix $A \in SU(2)$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longleftrightarrow \mathcal{F}_A(z) = \frac{az+b}{cz+d} \qquad a, b, c, d, z \in \mathbb{C}. \tag{3}$$

The Möbius transformations generate the conformal group in the plane and can be identified using stereographic projection with conformal transformations on the sphere. The action of

the Möbius group on the Riemann sphere is transitive in the sense that there is a unique Möbius transformation which takes any three distinct points on the Riemann sphere to any other set of three distinct points. Commutativity of the above diagram means that for any one-qubit state $|\psi\rangle$ and any $A \in SU(2)$ we have

$$\mathcal{F}_{A}\mathcal{P}(|\psi\rangle) = \mathcal{P}(A|\psi\rangle). \tag{4}$$

This shows that the conformal mapping $\mathcal{P}$ intertwines between any single qubit unitary operation A and its corresponding Möbius transformation $\mathcal{F}_{A}$. It is tempting to try to extend this diagram to the system of bipartite two-qubit systems. However due to the deference between the dimensions of single qubit and two-qubit systems, all the above processes must be modified in a convenient way which is the task of the next section.

3 Basic tools and definitions

We will require some preliminary definitions and results. Therefore this section devoted to provide some basic tools and background to attack to the geometrical properties of two-qubit pure states.

3.1 Quaternionic conformal map

The Hilbert space $\mathcal{H}_4^{\mathbb{C}}$ for the compound system is the tensor product of the individual Hilbert spaces $\mathcal{H}_2^{\mathbb{C}} \otimes \mathcal{H}_2^{\mathbb{C}}$ with a direct product basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. A two-qubit pure state reads

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \qquad \alpha, \beta, \gamma, \delta, \in \mathbb{C}, \tag{5}$$

with normalization condition $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$. Using quaternionic skew-field $\mathbb{Q}$ we can equivalently restate every $|\psi\rangle \in \mathcal{H}_4^{\mathbb{C}}$ by a quaterbit $|\widetilde{\psi}\rangle \in \mathcal{H}_2^{\mathbb{Q}}$ as [2]

$$\mathcal{Q}(|\psi\rangle) := |\tilde{\psi}\rangle = q_1|\tilde{0}\rangle + q_2|\tilde{1}\rangle \quad , \quad q_1 = \alpha + \beta\mathbf{j} \quad , \quad q_2 = \gamma + \delta\mathbf{j} \quad , \quad |q_1|^2 + |q_2|^2 = 1. \tag{6}$$

One can easily check that the map $\mathcal{Q}$ is a complex linear map that is

$$\mathcal{Q}(c_1|\psi_1\rangle + c_2|\psi_2\rangle) = c_1\mathcal{Q}(|\psi_1\rangle) + c_2\mathcal{Q}(|\psi_2\rangle) \qquad \forall\ c_1, c_2 \in \mathbb{C}.$$

The simplest way to introduce conformal map for two-qubit system is to proceed along the same line as for one-qubit case, but using quaternions instead of complex numbers (see appendix):

$$\mathcal{P}(|\tilde{\psi}\rangle) := q_1 q_2^{-1} = \frac{1}{|q_2|^2}[(\alpha + \beta\mathbf{j})(\overline{\gamma} - \delta\mathbf{j})] = \frac{1}{|q_2|^2}(S + C\mathbf{j}) \in \widetilde{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}, \tag{7}$$

where the Schmidt (S) and concurrence (C) terms are defined as follows

$$S := \alpha\overline{\gamma} + \beta\overline{\delta} \qquad , \quad C := \beta\gamma - \alpha\delta. \tag{8}$$

It should be mentioned that the map $\mathcal{P}$ is related to the projection of the second Hopf fibration of the form

$$\mathcal{P} : \mathbb{Q}^2 \longrightarrow \mathbb{Q}P^1 \tag{9}$$

where $\mathbb{Q}P^1$, is the one dimensional quaternionic projective space. Note that if $S = 0$ then $|\psi\rangle$ has Schmidt decomposition

$$|\psi\rangle = |q_1||0\rangle_1|e\rangle_2 + |q_2||1\rangle_1|f\rangle_2, \tag{10}$$

where $\{|e\rangle, |f\rangle\}$ is two orthonormal basis for second qubit. Moreover C is proportional to one of entanglement measure $\mathcal{C}(|\psi\rangle) := \langle\psi|\sigma_y \otimes \sigma_y|\overline{\psi}\rangle$ called *concurrence* [11] where $\overline{\psi}$ denotes the complex conjugation and σ_y is one of Pauli spin operators. Concurrence is widely used to quantify entanglement of two-qubit systems. In fact $2\overline{C} = \mathcal{C}$ and if $\mathcal{C} = 0$ then $|\psi\rangle$ unentangled in the sense that it can be written as a tensor product of two pure state of individual subsystems, i.e., $|\psi\rangle = |\phi\rangle_1|\varphi\rangle_2$. The quaternionic conformal map $\mathcal{P}'$ defined by $\mathcal{P}' := q_2^{-1}q_1$ is distinct from $\mathcal{P}$ and may be interpreted as one for dual space. Indeed it can be easily verify that $\mathcal{P}'(\langle\tilde{\psi}|) = \overline{\mathcal{P}(|\tilde{\psi}\rangle)}$.

3.2 Local unitary subgroup of $Sp(2)$

Before discussing the local unitary subgroup of $Sp(2)$ we would add some short discussion on the transformation properties of the two-qubit entangled state and its quaternionic representative. It would simplify later presentation considerably if we represent the $|\psi\rangle$ of Eq. (5) by the 2×2 matrix

$$\Psi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{11}$$

which gives rise to the transformation property $|\psi\rangle \to A' \otimes A|\psi\rangle$, for $A', A \in SU(2)$

$$\Psi \mapsto A'\Psi A^T \tag{12}$$

where A^T refers to the transposed of A. The quaternionic version of this transformation is

$$|\tilde{\psi}\rangle \to A'|\tilde{\psi}\rangle(a - \bar{b}\mathbf{j}). \tag{13}$$

The quaternionic counterpart of group $SU(2)$ for two-qubit system seems to be group $Sp(2)$ which is defined as

$$Sp(2) := \{B \in GL(2,\mathbb{Q}) : B^\dagger B = I\}, \tag{14}$$

or equivalently can be expressed by

$$Sp(2) := \{U \in U(4) : UJU^T = J\}. \tag{15}$$

where $J := I \otimes \varepsilon$, with $\varepsilon \equiv -i\sigma_2$. Since the two-qubits systems have entanglement property therefore we will consider operations which do not change the entanglement measure (concurrence) throughout the diagram. As it is well known such operations must act locally on each individual qubit . Therefore we restrict ourself to local subgroup $\mathcal{B} \simeq SO(2) \otimes SU(2)$ of group $Sp(2)$ where its corresponding complex form $\mathbb{C}B$ reads

$$\mathbb{C}B = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \otimes \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \tag{16}$$

and we will investigate the problem for this local unitary operations in the next section. Within this interesting scenario it is a trivial and well-known fact that the measure of entanglement (concurrence) does not change regarding the local unitary transformations like $\mathcal{B}$.

3.3 Quaternionic Möbius transformations

The main difficulties in establishing the quaternionic approach to Möbius transformation is the non-commutativity of the quaternions. Beside that, it is rather seamless to carry over much of the complex theory. For $M \in SL(2,\mathbb{Q})$ we define the QMT

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \longleftrightarrow \mathcal{F}_{_M}(q) := (q\ m_{11} + m_{12})(q\ m_{21} + m_{22})^{-1} \qquad m_{ij} \in \mathbb{Q}, \tag{17}$$

with the conventions $\mathcal{F}_M(\infty) = m_{11}m_{21}^{-1}$ and $\mathcal{F}_M(-m_{22}m_{21}^{-1}) = \infty$. As is the case with $\tilde{\mathbb{C}}$, there is a Möbius transformation taking any three given points to any other three points, however, it is not unique. It is easily seen that $\mathcal{F}_{_{MM'}} = \mathcal{F}_{_M} \circ \mathcal{F}_{_{M'}}$ where matrix multiplication is defined in usual way, i.e., $MM' = (m_{i1}m'_{1j} + m_{i2}m'_{2j})$. The set of all such transformations forms a group under composition. This set is identified naturally with the quotient space $PSL(2,\mathbb{Q}) = SL(2,\mathbb{Q})/\{\pm I\}$. However, again due to the non-commutativity of the field $\mathbb{Q}$, in addition to $\mathcal{F}_{_M}$ there are several possibilities to define the QMT [16], i.e.,

$$\begin{aligned} \mathcal{F}'_{_M}(q) &= (q\ m_{21} + m_{22})^{-1}(q\ m_{11} + m_{12}) \\ \mathcal{F}''_{_M}(q) &= (m_{11}\ q + m_{12})(m_{21}\ q + m_{22})^{-1} \\ &\vdots \end{aligned}$$

Therefore the extending of the commuting QCM which intertwines between the group $\mathcal{B}$ and the corresponding QMT fixing the measure of entanglement is our purpose. As we will see in the next section to attribute physical interpretation for the QMT and to fit the problem in a commutative setting it is necessary to choose Eq.(17) as a preferable definition of QMT. This choice for the QMT is based on the implicit fact that we treat the space of quaternionic spinors as a right module (multiplication by scalars from the right).

4 Two-qubit geometry

We now proceed one step further, and investigate the results of the previous section for two-qubit pure states under the action of local unitary subgroup $\mathcal{B}$ of $Sp(2)$.

In this section it will be shown that a direct substitution of the definitions of previous section leads to the commutativity of the following diagram

$$\begin{array}{ccccc}
\mathcal{H}^{\mathbb{C}}_{2\otimes 2} & \xrightarrow{\mathcal{Q}} & \mathcal{H}^{\mathbb{Q}}_{2} & \xrightarrow{\mathcal{P}} & \widetilde{\mathbb{Q}} \\
{\scriptstyle \mathbb{C}B}\downarrow & & \downarrow{\scriptstyle B} & & \downarrow{\scriptstyle \mathcal{F}_B} \\
\mathcal{H}^{\mathbb{C}}_{2\otimes 2} & \xrightarrow{\mathcal{Q}} & \mathcal{H}^{\mathbb{Q}}_{2} & \xrightarrow{\mathcal{P}} & \widetilde{\mathbb{Q}}
\end{array}$$

The purpose of the diagram is to verify that wether the QCM intertwines between the operator $B \in \mathcal{B}$ and the corresponding QMT $\mathcal{F}_{B}$. This implies that for any two-qubit pure state we expect that the following equalities

$$\mathcal{P}\mathcal{Q}(\mathbb{C}B|\psi\rangle) =^{?} \mathcal{P}B(\mathcal{Q}|\psi\rangle) =^{?} \mathcal{F}_{B}\mathcal{P}(\mathcal{Q}|\psi\rangle), \tag{18}$$

hold for any $|\psi\rangle \in \mathcal{H}^{\mathbb{C}}_{4}$. By choosing of each equality above one can breakdown this diagram into three pairs of commutative pieces. Therefore we study each of them which every two-qubit (quaterbit) can be influenced by the maps introduced above. The above equalities follow from the following calculations.

4.1 Calculating $\mathcal{P}\mathcal{Q}(\mathbb{C}B|\psi\rangle)$

It is convenient to start with the first statement in Eq.(18)

$$\mathcal{P}\mathcal{Q}(\mathbb{C}B|\psi\rangle) = \mathcal{P}(|\tilde{\psi}'\rangle) = \mathcal{P}(q'_1|\tilde{0}\rangle + q'_2|\tilde{1}\rangle) = q'_1 q'^{-1}_2, \tag{19}$$

where q'_1 and q'_2 are results of the action of Eq. (16) on the general two-qubit pure state Eq. (5) followed by the map $\mathcal{Q}$ as

$$q'_1 = \alpha' + \beta'\mathbf{j} = [(a\alpha + b\beta)\cos\theta + (a\gamma + b\delta)\sin\theta] + [(\bar{a}\beta - \bar{b}\alpha)\cos\theta + (\bar{a}\delta - \bar{b}\gamma)\sin\theta]\mathbf{j}\ ,$$

$$q_2' = \gamma' + \delta'\mathbf{j} = [(a\gamma + b\delta)\cos\theta - (a\alpha + b\beta)\sin\theta] + [(\bar{a}\delta - \bar{b}\gamma)\cos\theta - (\bar{a}\beta - \bar{b}\alpha)\sin\theta]\mathbf{j}\ . \quad (20)$$

On the other hand the Eq. (19) can be expressed in terms of Schmidt and concurrence terms

$$\mathcal{PQ}(\mathbb{C}B|\psi\rangle) = \frac{1}{|q_2'|^2}(S' + C'\mathbf{j}), \quad (21)$$

where the norm of q_2' is

$$|q_2'|^2 = |q_2|^2\cos^2\theta + |q_1|^2\sin^2\theta - \sin 2\theta\ \mathrm{Re(S)},$$

and the S' and C' are given by

$$S' = \alpha'\overline{\gamma'} + \beta'\overline{\delta'} = \cos^2\theta\ S - \sin^2\theta\ \bar{S} + \frac{1}{2}\sin 2\theta(|q_2|^2 - |q_1|^2)\ , \quad (22)$$

$$C' = \beta'\gamma' - \alpha'\delta' = C. \quad (23)$$

We observe that independent of the parameters a, b and θ, the concurrence term C' is invariant under the action of $\mathbb{C}B$. This observation is well known and fulfills our expectations that entanglement can be changed only by global transformations. An interesting situation arises when $\theta = 0$ e.g., $S' = S$. This case is coincide with the results of Mosseri et al in [2].

This is not the only way to get above results dealing with the geometry of two-qubit states. In the next subsection we shall establish the similar operations on a quaterbit and find the same results.

4.2 Calculating $\mathcal{P}B(\mathcal{Q}|\psi\rangle)$

Considering the diagram we can proceed another approach to understand more about the two-qubit entangled state. Unlike in the definition of $\mathcal{B}$ in order to correctly represent the complex transformation on the two-qubit state, the separable $Sp(2)$ transformation on the quaternionic spinor should be represented by left action of the 2×2 matrix $A' \in SO(2)$ containing $\sin\theta$ and $\cos\theta$, and right multiplication with the quaternion $a - \overline{b}\mathbf{j}$, as in Eq. (13). Hence by applying

the $B \in \mathcal{B}$ on a quaterbit $|\tilde{\psi}\rangle$ one can get

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} |\tilde{\psi}\rangle(a - \bar{b}\mathbf{j}) = \begin{pmatrix} q'_1 \\ q'_2 \end{pmatrix},$$

where q'_1 and q'_2 are the same as in Eq. (20). Therefore the relevant part of the crucial diagram (the first quadrangle) is commutative, i.e.,

$$\mathcal{Q}(\mathbb{C}B|\psi\rangle) = B(\mathcal{Q}|\psi\rangle). \tag{24}$$

It is clear that applying the QCM on the both side of the above equation lead to the first equality in Eq. (18).

4.3 Calculating $\mathcal{F}_{B}\mathcal{P}(\mathcal{Q}|\psi\rangle)$

We have already shown that the first equality in Eq. (18) holds. Let us now see what happen on a quaterbit regarding the action of QMT. Using the linear map $\mathcal{Q}$ together with QCM on a two-qubit pure state in Eq. (5) yield

$$\mathcal{P}\mathcal{Q}(|\psi\rangle) = \mathcal{P}(|\tilde{\psi}\rangle) = \mathcal{P}(q_1|\tilde{0}\rangle + q_2|\tilde{1}\rangle) = q_1 q_2^{-1} = \frac{1}{|q_2|^2}(S + C\mathbf{j}) \tag{25}$$

Furthermore this point is mapped under the action of the QMT in Eq. (17) as follows

$$\begin{aligned} \mathcal{F}_{B}\left(\frac{1}{|q_2|^2}(S + C\mathbf{j})\right) &= \left(\frac{1}{|q_2|^2}(S + C\mathbf{j})(a - b\mathbf{j})\cos\theta + (a - b\mathbf{j})\sin\theta\right) \\ &\quad \times \left(\frac{1}{|q_2|^2}(S + C\mathbf{j})(-a + b\mathbf{j})\sin\theta + (a - b\mathbf{j})\cos\theta\right)^{-1} \\ &= \frac{\cos^2\theta\ S - \sin^2\theta\ \bar{S} + \sin\theta\cos\theta(|q_2|^2 - |q_1|^2)\ + C\ \mathbf{j}}{|q_2|^2\cos^2\theta + |q_1|^2\sin^2\theta - \sin 2\theta\ \mathrm{Re(S)}} \end{aligned}$$

Again we get precisely the same result as the two previous subsections meaning that the second equality in (18) holds. This in turn implies that the QCM intertwines between the operator $B \in \mathcal{B}$ and the corresponding QMT $\mathcal{F}_{B}$. This is what we have expected to see. To sum up we have the three commutative diagrams for two-qubit pure states as mentioned above.

4.4 $SU(2)\otimes SO(2)$ transformation

So far we have considered the $SO(2)\otimes SU(2)$ subgroup of $Sp(2)$ and find the total commutative diagram. Let us now see what has been gained in considering the separable subgroup $\mathcal{B}'\simeq SU(2)\otimes SO(2)$, in the sense that $SU(2)$ and $SO(2)$ act on the first and second particles of the pure two-qubit state respectively. It is easy to show that for this to occur, one must consider the map

$$M \longrightarrow \mathbb{C}M : M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \longrightarrow \left(\begin{array}{cc|cc} z_{11}^{(1)} & z_{12}^{(1)} & -\bar{z_{11}}^{(2)} & -\bar{z_{12}}^{(2)} \\ z_{21}^{(1)} & z_{22}^{(1)} & -\bar{z_{21}}^{(2)} & -\bar{z_{22}}^{(2)} \\ \hline z_{11}^{(2)} & z_{12}^{(2)} & \bar{z_{11}}^{(1)} & \bar{z_{12}}^{(1)} \\ z_{21}^{(2)} & z_{22}^{(2)} & \bar{z_{21}}^{(1)} & \bar{z_{22}}^{(1)} \end{array}\right),$$

where $m_{ij}=z_{ij}^{(1)}+z_{ij}^{(2)}\mathbf{j}$ and $z_{ij}^{(1)},z_{ij}^{(2)}\in\mathbb{C}$, which in turn induces the following definition for 2×2 symplectic group

$$Sp(2) := \{U\in U(4) : U^T J' U = J'\}. \tag{26}$$

where in this case $J'=\varepsilon\otimes I$. Note that in the transformation $A'\otimes A|\psi\rangle$ and subsequently in its matrix form Eq. (12), A' and A could be any member of group $SU(2)$. On the other hand in the quternionic version of transformation $SO(2)\otimes SU(2)$ on a quaterbit we were not worry about left or right action of the $A'\in SO(2)$ on a quaternionic spinor. However here in our discussion $A'\in SU(2)$ while $A\in SO(2)$ and the former acts on the quaternionic spinor. Therefore unlike the $SO(2)\otimes SU(2)$ case, one must distinguish between the left and right actions regarding quaternionic version of the separable subgroup $SU(2)\otimes SO(2)$ on a quaterbit $|\widetilde{\psi}\rangle\in\mathcal{H}_2^{\mathbb{Q}}$. Roughly speaking in this case we must use the left action as follows

$$\begin{pmatrix} aq_1+bq_2 \\ -\bar{b}q_1+\bar{a}q_2 \end{pmatrix}(\cos\theta-\sin\theta\mathbf{j}) = \begin{pmatrix} q_1' \\ q_2' \end{pmatrix}$$

which implies that the first quadrangle in the main diagram is commutative, i.e.,

$$\mathcal{Q}(\mathbb{C}B'|\psi\rangle) = B'(\mathcal{Q}|\psi\rangle), \tag{27}$$

where $B' \in \mathcal{B}'$. But unfortunately, in this case there is no apparent way to pick a particular QMT in order to get total commutative diagram and hence the problem of intertwining QCM will cease to exist.

5 Conclusion

In this paper we considered the action of $SO(2) \otimes SU(2)$ part of quaternionic group $Sp(2)$ on a two-qubit pure state as a local transformation which obviously leaves invariant the measure of entanglement. We have shown that QCM intertwines between local unitary subgroup $Sp(2)$ and QMT. It is rather interesting that the three way are so well related to the important ingredients of a pure two-qubit state which are Schmidt and concurrence terms. In this investigation we found that other definitions for QMT do not work. Another simple consequence of our findings is that the choice of $SU(2)$ action on the first particle leads to the some essential changes on the main diagram in the sense that just the first quadrangle become commutative together with the fact that one have to use the left action on the quaterbit. Moreover in using $SU(2) \otimes SO(2)$ on the pure two-qubit state, there is no QMT which make the diagram total commutative and subsequently there is nothing to do with QCM.

Appendix: Quaternion

The quaternion skew-field $\mathbb{Q}$ is an associative algebra of rank 4 over $\mathbb{R}$ whose every element can be written as

$$q = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \qquad , \quad x_0, x_1, x_2, x_3 \in \mathbb{R} \quad \text{with} \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

It can also be defined equivalently, using the complex numbers $z_1 = x_0 + x_1\mathbf{i}$ and $z_2 = x_2 + x_3\mathbf{i}$ in the form $q = z_1 + z_2\mathbf{j}$ endowed with an involutory antiautomorphism (conjugation) such as

$$q = z_1 + z_2\mathbf{j} \in \mathbb{C} \oplus \mathbb{C}\mathbf{j} \;\longrightarrow\; \bar{q} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k} = \bar{z}_1 - z_2\mathbf{j}.$$

Every non-zero quaternion is invertible, and the unique inverse is given by $q^{-1} = \frac{1}{|q|^2}\bar{q}$ where the quaternionic norm $|q|$ is defined by $|q|^2 = q\bar{q} = |z_1|^2 + |z_2|^2$. The norm of two quaternions q and p satisfies $|qp| = |pq| = |p||q|$. Note that quaternion multiplication is non-commutative so that $\overline{q_1 q_2} = \overline{q_2}\,\overline{q_1}$ and $\mathbf{j}z = \bar{z}\mathbf{j}$, where the last relation have been used in this paper extensively. On the other hand a two dimensional quaternionic vector space V defines a four dimensional complex vector space $\mathbb{C}V$ by forgetting scalar multiplication by non-complex quaternions (i.e., those involving $\mathbf{j}$ or $\mathbf{k}$). Roughly speaking if V has quaternionic dimension 2, with basis $\{|\widetilde{0}\rangle, |\widetilde{1}\rangle\}$, then $\mathbb{C}V$ has complex dimension 4, with basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Moreover each matrix $M \in M(2, \mathbb{Q})$, i.e., each linear map $M = (m_{ij}) : V \longrightarrow V$ defines a linear map $\mathbb{C}M : \mathbb{C}V \longrightarrow \mathbb{C}V$ i.e., a matrix $\mathbb{C}M \in M(4, \mathbb{C})$. Concretely, in passing from V to $\mathbb{C}V$ each entry $m_{ij} = z_{ij}^{(1)} + z_{ij}^{(2)}\mathbf{j}$ is replaced by 2×2 complex matrix which means that the map

$$M \longrightarrow \mathbb{C}M : M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \longrightarrow \left(\begin{array}{cc|cc} z_{11}^{(1)} & -z_{11}^{(2)} & z_{12}^{(1)} & -z_{12}^{(2)} \\ \bar{z}_{11}^{(2)} & \bar{z}_{11}^{(1)} & \bar{z}_{12}^{(2)} & \bar{z}_{12}^{(1)} \\ \hline z_{21}^{(1)} & -z_{21}^{(2)} & z_{22}^{(1)} & -z_{22}^{(2)} \\ \bar{z}_{21}^{(2)} & \bar{z}_{21}^{(1)} & \bar{z}_{22}^{(2)} & \bar{z}_{22}^{(1)} \end{array}\right),$$

is injective and it preserves the algebraic structures such as $\mathbb{C}(M + M') = \mathbb{C}M + \mathbb{C}M'$, $\mathbb{C}(MM') = (\mathbb{C}M)(\mathbb{C}M')$ and $\mathbb{C}(M^\dagger) = (\mathbb{C}M)^\dagger$, where $(M^\dagger)_{ij} = \bar{m_{ji}}$. It is easy to see that

$$M(2, \mathbb{Q}) = \{M \in M(4, \mathbb{C}) : JMJ^{-1} = \bar{M}\},$$

where metric J is given by

$$J := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Considering $GL(2, \mathbb{Q}) \subseteq M(2, \mathbb{Q})$ for the subset of invertible matrices; it is well known [14, 15, 13] that M has a two-sided inverse in $M(2, \mathbb{Q})$ if and only if the $\mathbb{C}(M(2, \mathbb{Q}))$ is invertible in

$M(4,\mathbb{C})$ which implies that $\mathbb{C}(M(2,\mathbb{Q}))$ belongs to the group $GL(4,\mathbb{C})$ which consisting of all invertible 4×4 matrices. Of course, in this description, we also have

$$GL(2,\mathbb{Q}) = \{M \in GL(4,\mathbb{C}):\ JMJ^{-1} = \bar{M}\},$$

$$SL(2,\mathbb{Q}) = \{M \in GL(4,\mathbb{C}):\ \det M = 1\ ,\ JMJ^{-1} = \bar{M}\}.$$

Acknowledgments

It is a pleasure for one of the authors (GN) to acknowledge enlightening discussions about quaternion conformal map with H. Fakhri. We also acknowledge private communications with R. Sufiani for which we are deeply grateful. The authors also acknowledge the support from the Mohaghegh Ardabili University.

References

[1] J. Lee, C. H. Kim, E. K. Lee, J. Kim and S. Lee, Quantum Information Processing,Vol.**1**, Nos.1/2,p 129, (April 2002).

[2] R. Mosseri and R. Dandoloff, J. Phys. A: Math. Gen. **34**, 10243 (2001).

[3] P. Lévay, J. Phys. A: Math. Gen. **37**, 1821 (2004).

[4] P. Lévay, Phys. Rev. A **71**, 0123434 (2005).

[5] P. Lévay, J. Phys. A: Math. Gen. **39**, 9533 (2006).

[6] B. A. Bernevig and H. D. Chen, J. Phys. A: Math. Gen. **36**, 8325 (2003).

[7] H. Heydari, Quantum Information and Computation, **6**, 400 (2006).

[8] H. Heydari, J. Phys. A: Math. Gen. **39**, 9839 (2006).

[9] B. M. Terhal, M. Horodecki, D. W. Leung, D. P. DiVincenzo, J. Math. Phys. **43**, 4286 (2002).

[10] M. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. D. Sen, U. Sen, B. Synak Phys. Rev. A **71**, 062307 (2005).

[11] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998); S. Hill and W. K. Wootters, ibid. **78**, 5022(1997).

[12] P. Di Francesco, P. Mathieu, D. Sénéchal, *Conformal Field Theory*, Springer (1996).

[13] S. D. Leo, G. C. Ducati, International Journal of Theoretical Physics, Vol. **38**, No. 8, (1999).

[14] H. Aslaken, Math. Intelligencer **18**, (1996), 57-65. MR **97j**:16028.

[15] F. R. Harvey, *Spinors and Calbrations*, Academic Press (1990). MR **91e**:53056.

[16] R. M. Porter, Complex Variables Theory and Applications, **36**, 285 (1998).

[17] S. L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields* (Oxford UP, New York, 1995).

[18] P. Ginsparg, "*Applied conformal field theories*" in LES HOUCHES Lecture, edited by E. Brezin and J. Zinn-Justin (1988).

[19] J. M. Fernandez, W. A. Schneeberger, arXiv:quant-ph/0307017 (2003).

[20] S. D. Leo, G. Scolarici, J. Phys. A: Math. Gen. **33** , 2971 (2000).